The triple nature of mathematics: deep ideas, surface representations, formal models

Zbigniew Semadeni
Warsaw University, Poland

Introduction. At ICME-2 in Exeter, 32 years ago, I listened to the plenary lecture delivered by René Thom (1972). His apt analysis clearly demonstrated the untenability of the basic assumptions (mathematical and philosophical) of the "modern mathematics" movement (i.e., "new math"), which had just reached its zenith. The promoters of the changes kept advocating the formalist approach to school mathematics. They overemphasized the role of propositional calculus, of set theory, of general structures (algebraic, topological), and believed in the effectiveness of explicitly naming the basic properties of operations (commutativity etc.) in the computations performed by students. The implementation of their ideas resulted in premature, unnecessary abstraction. New math adherents insisted on axiomatic systems, on proofs and rigour, called for abandoning the Euclidean geometry, and treated applications (to physics and to problems of everyday life) as irrelevant. They also pointedly neglected meaning in mathematics and were preoccupied with its syntax. I remember that several participants, profoundly shocked by Thom's arguments against new math, considered his position as untenable (in spite of his prestige as a Field's medalist).

This paper[*] owes much to Thom's. It concerns *the epistemology of mathematics*. Let us recall that epistemology is a major branch of philosophy which may be described as the theory of cognition, the study of the origin, nature, methods, validity and limits of scientific knowledge. A comprehensive survey of main problems of various epistemologies of mathematics and of mathematics education is given in (Sierpińska and Lerman, 1996).

It should be pointed out that *reflections on the nature of mathematics are important because its image* (fixed in mathematicians' minds) *is conveyed to educators and prospective teachers and then influences the education* (curricula, textbooks, classroom practice) *in a direct way*. The failures of "new math" (whose epistemology was reduced to axiomatic systems) have shown this in a persuasive way. Since that time, however, the pendulum has swung back. The past two decades have witnessed another extreme: *the postmodernist tendency of renouncing crucial attributes of mathematics by some philosophers* and their followers who regard theorems as "social constructs" and view the truth of a theorem as a "social consensus", subject to negotiations and change, and the negation of a theorem as, possibly, "an alternative viewpoint", much as if mathematics were a social science. There are only conceptions and no misconceptions. Radical ultrarelativist "isms" deny the very possibility of objective truth, knowledge and validity; hence their statements are nonfalsifiable (Freudenthal, 1991, p.146; Goldin, 2002; Goldin, 2003). If such a theoretical "paradigm" is unable to provide an adequate explanation of some fundamental questions, then its protagonists simply evade the problem by declaring that it is irrelevant or that it is only a matter of language, of social convention etc. Claims of this sort attract favour of those educators who oppose the closed-minded, "absolutists" views of mathematicians (they also fit well into the bitter experience of those myriads of people who took behaviourist courses in mathematics and were never able to remember and correctly use the tangle of rules imposed by teachers). On the other hand, some of postmodernist ideas have a sound core and support positive, highly significant educational changes, e.g., an attitude of tolerance and openness towards visually-oriented activities, discovery process, open-ended problem solving, students' thinking and conceptions, real-

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life context-embedded learning. A vital question is at what point this openness becomes permissiveness and tolerance of serious errors; the line between them is thin. *The research paradigms of mathematics education seem to be drifting away from the paradigm of mathematics itself* (Sfard, 1998). Moreover, there is a serious widening deplorable gulf between the philosophy of research mathematicians (respectively, scientists) and the philosophy of philosophers dealing with mathematics (respectively, natural sciences). The long-term effect of these trends on the education is to be judged by the next generation.

The nature of mathematics is often presented in terms of a duality. The best-known "*pure mathematics"* versus "*applied mathematics"* goes back to ancient times (Plato versus Archytas of Tarentum). With a shift of emphasis, this may be presented in the form: "*mathematics as a theory"* versus "*mathematics as a set of useful tools and competencies"*. Towards the end of 20th century, still another issue gained widespread popularity: "*Mathematics as a body of abstract, formal, absolute, sure, eternal knowledge"* versus "*mathematics as human activities, problem solving, discourse"*. This list of contrasting features of mathematics can be continued. It is not uncommon to consider such a pair, wrongly, as a dichotomy (e.g., some people draw the invalid conclusion: mathematics is a product of human activity, therefore its theorems are as fallible as other products of human thought). Although the features of mathematics articulated in each of these three dual pairs are naturally set in opposition to each other, we would like to stress that they should not be regarded as describing an either/or situation. They are not mutually exclusive; rather, they reinforce each other. In particular, mathematics has two "faces": one *pure*, very abstract, and another *applied*, "unreasonably effective in the natural sciences" (Wigner, 1960). There are an astonishing number of cases where ideas and theorems developed within pure mathematics (with no applications in mind) later found unexpected, highly successful applications in science and technology. This phenomenon is clearly incompatible with postmodernist claims.

The triad: **«deep, surface, formal».**

The core of this paper is a proposal that the above conceptions expressed in terms of certain dualities should be enriched by a quite different conception of the **triple nature of mathematics**, namely we will argue for distinguishing: **deep** ideas, **surface** representations and **formal** models of various mathematical objects[*].

Two elements of this triad have their origins in psycholinguistics (the Chomsky theory). Inspired by this theory, Thom (1972) pointed out that the domain of logic and propositional calculus includes only the "crudest joints" of our reasoning, representing its most superficial aspects, corresponding to the surface structures of linguistics. These crude joints neglect the fine interactions due to sense, which are difficult to explain or formalize. Later Richard Skemp (1982) called for drawing a distinction between the *surface structures* (syntax of the mathematical symbol-system) and the *deep structures* (semantics), pointing out that the meaning of a mathematical communication lies in the deep structures. Deep structures are of key importance, but are not accessible to people. Only surface structures can be transmitted. Acting on these hints, we develop the conception of the first two elements of the triad; nevertheless, since the word "structure" has different and well-established meanings in mathematics, we replace it by "idea" and "representation", respectively. Admittedly, in this way we lose an advantage offered by the word "structure", which connotes the structural aspects of those abstract entities. *When we deal with deep ideas and surface representations we should bear in mind their systemic nature*. Surface

[*] We shorten the descriptions by using the auxiliary term: **mathematical object**. It may stand for a concept, relation, proposition, propositional function, theorem, proof, a piece of reasoning, an algorithm, a subroutine etc. Conceivably, any such object could be considered as an element of a suitable set.

representations are not separate symbols; they are parts of various heterogeneous systems. Deep ideas also form intricate webs, which are difficult to analyse.

Formal models. The triad is completed when the distinct character of mathematics is taken into consideration. The two elements derived from linguistics are augmented with a third one. By a **formal model** of a mathematical object we understand any of its counterparts in an **axiomatic theory**. This element of the triad is best known and described in many books. Therefore we discuss only its relations to the other elements of the triad. Although we set out several arguments to the effect that the deep ideas prevail in the triad, the formal models are an indispensable part of theoretical mathematics, crucial for research and also for certain applications*. Nevertheless, they may play a negative role in education if they are regarded as model examples of proper reasoning.

Surface representations of a mathematical object are **signs** (which can be seen, heard, touched, manipulated) **representing** this object. Typically they consist of words (spoken or written) as well as of various marks and drawings on paper, blackboard, and screen or in computer memory, but we broaden the scope of the concept by including gestures (expressing mathematical ideas through a motion), wooden models of solids, sets of counters (representing numbers), spacial patterns, and the like. This list includes symbols that are subject to strict syntax rules of mathematical symbol-systems as well as symbols that admit more flexible interpretations (examples: explaining properties of a solid by pointing out the edges of a model; using kinetic depictions; graph-supported visual arguments). Surface representations are essentially the same as *Darstellungen* (external representations, Meissner, 2002). They serve various purposes:

1. They are **means to communicate** mathematical thoughts, ideas, reasoning etc. *to other people.* They serve as an interface between the inner world of thought and the outer world. Each of these representations has a *dual status*: it is a *physical thing* (a sound, a piece of chalk, a bodily movement) which can be *perceived by senses* and at the same time it is a *mental object*, serving both as a “label” and as a “handle” with mathematical ideas attached to it. The representation entails an interpretation of what is perceived; is a means of “mediation” between concepts/thoughts represented by it and something physical.

2. Surface representations are indispensable **tools for working mathematically** (in computations, problem solving, proving). They somehow structure the way we conceive of mathematics. In particular, the human linguistic facility is essential for thinking. The symbol-system of *arithmetic, algebra, trigonometry, and calculus* is a powerful *tool for surface reasoning*, part of which can be done mechanically, when transformations of formal expressions lead to a result, which is to be scrutinized, read off and interpreted in terms of the situation in question. This may be used to produce new information from the given (e.g., the solution of an equation, a proof). The role of visual perception, symbol manipulation and observation is essential in the process of transforming formulas (which complements mental reasoning), and so is the role of habits related to details of notation (note, e.g., the difference between n^x and x^n). The power of external inscriptions and of diagrammatical reasoning is stressed by Peirce (1955) and Dörfler (2004).

3. Words and symbols may be **names** or **labels** of mathematical objects, and are thus **instrumental in forming abstract concepts**. For instance, the word “seventeen” and the symbol “17” are needed to separate this number from other numbers and to create a single concept, that of number 17, while the term “Banach space” helps to form a higher-order

* Models should not be confused with metaphors (unless one extends the scope of the latter so as to include everything). Although metaphors play a significant role in mathematical discourse, a formal model of X is not a metaphor of X (analogously, an architect's model of a house is a model and not a metaphor of a house).

concept of an object of functional analysis. Moreover, the similarity of certain symbolic representations may help to call attention to important analogies.

Deep ideas. The construct "deep idea" cannot be defined in simple analytic terms. At this point a preliminary description can be given: the *deep idea* of a mathematical object is a *well-formed abstract idea* which includes the *meaning* of the object, its *properties*, its *relationships* with other objects, both mathematical and non-mathematical, in real life and physics (its "conceptual domain", which reflects the experience with this object), and its *purposes* (that is, the reasons why this object is used and studied). However, the deep idea is not simply a sum of such constituent parts; it can become *mature, firm and flexible* as a result of some kind of a gradual "deep mental synthesis". We elaborate on these points below and explain some intrinsic questions concerning the proposed construct.

It should be stressed that our triad describes certain features of *mathematics as a body of the present human knowledge*. The theory is not meant to embrace the whole field of human activities that may be regarded as genuinely mathematical. In particular, we do not deal with such significant questions as the process of discovery, heuristics, problem solving, learning new ideas, applying mathematics to problems of the real world, although *they are crucial to the process of forming deep ideas*. In other words, we do not deal with what Freudenthal (1991, p.14) called "mathematics as an activity", although we regard it as a very important aspect of mathematics and the triad is meant to be helpful in its study.

Examples. "Deep idea" should be regarded as *a primitive notion*, explained in the context of the actual work on mathematics, by analysing pertinent examples and specially chosen situations. Sixteen basic examples have been selected and provided with comments that highlight significant relations between elements of the triad, their features and limitations (the remaining, unnumbered examples appear sporadically in various parts of the text).

FIRST EXAMPLE. The **expression** $9 + 24 = 33$ is a surface representation of a mathematical fact, which is represented by the symbols "9", "2", "4", "3", "+", "=". The corresponding deep idea is a (broadly understood) *meaning* of $9 + 24 = 33$ in various contexts (in real life or mathematics), links with related statements, and possible *purposes* for which this fact may be used. A formal model of the equality $9 + 24 = 33$ is a *true proposition* corresponding to it, expressed in the language of an axiomatic theory (e.g., in the Peano axiom system or in any axiom system of set theory).

Further typical examples are: *the deep idea of a particular concept* (e.g., of a specific number, say 24, of "negative number" in general, of "point", "triangle", "geometric figure", "cosine", "derivative", "stochastic independence", "quotient group"), the deep idea of a specific *theorem* (e.g., of the theorem of Pythagoras) and of a *proof* of a specific theorem (to be distinguished from the deep ideas of the general concepts: *theorem* and *proof* in mathematics), the deep idea of a specific mathematical *procedure* (e.g., of solving an equation of the given type) or an *algorithm*. We may also consider the deep ideas of typical objects at much higher levels of abstraction, such as groups, equationably definable classes of algebras, categories (Example 14). In order to avoid lengthy sentences, instead of "*the deep idea of an object* X" we may simply say "*the deep idea* X", e.g. the deep ideas "power series", "the equality $(a + b)^2=a^2 + 2ab + b^2$". The phrase "X *is a deep idea*" means that it makes sense to speak of the deep idea of X; for instance, we may say that Euler's identity $e^{i\pi} = -1$ is a single deep idea, and so, too, are reduction to the lowest common denominator and the algorithm of long division.

Deep ideas have a *dual status*: **psychological** (mental objects) and **epistemological**. We distinguish between "*individual deep ideas* of X" in minds of various persons and "*the deep idea* of X", which is a *single abstract epistemological object*, an idealized common abstract version. The latter must have some *permanent intersubjective content* (Examples 1-16 show how to interpret this). On the other hand, the individual deep ideas of X of different people need not be identical. They are *purely mental objects, invisible and inaudible* (sometimes they are not easily accessible even to their possessors). They *can be communicated to other people only by surface representations*, that is, by words, symbols, drawings, gestures etc. We may speak of an individual deep idea of X when it is *sufficiently well formed in the mind* of the given person, and this presupposes: (a) the presence of a feeling of familiarity with the object, (b) a sense of correctness of certain basic statements concerning X, (c) *adequate understanding* of X, (d) *robustness of understanding in cases of typical cognitive conflicts*. Hints of how to interpret the requirements (c) + (d) (and how one can judge whether they are satisfied) are scattered throughout the paper. It should be emphasized that perceptually justified knowledge, mental imaging or surface-verified proofs do not by themselves yield deep ideas.

SECOND EXAMPLE. The celebrated conception by Piaget of the so-called **conservation of the** (cardinal) **number** means that a child, at some stage of mental development, becomes deeply convinced that *the cardinality of the set* consisting of, say, 10 apples *does not change when* the apples *are spread out* so they cover a larger area (Piaget and Inhelder, 1989). *When it becomes stable, context-independent and applies to any number of physical objects, the conservation becomes a deep idea*. Such an invariance of the cardinal number of moving objects can hardly be proved formally, because any proof would involve a mathematization of the situation in the language of set theory, and then one would face the problem of proving the correctness of the passage from reality to a formal model. In contrast, the conservation underlies a multitude of deep ideas and mathematizations. (Actually, one may think of two aspects of the conservation: 1. conviction that after spreading out *the set remains the same*, 2. conviction that *if it is the same set then it must contain the same number of elements*; yet, the formation of the deep idea of conservation does not require that the person have an explicit concept of a set.)

THIRD EXAMPLE. The deep idea "**number pi**" is a single mental idea. It includes the definition of π, the situations in which π is used, and the sense of π in various contexts (often divorced from geometry). However, if a rigorous definition of π is required, we have to use a theory of real numbers. We may choose, e.g., π in Cantor's theory (let us denote this set by π_C), but we may just as well choose π_D in Dedekind's theory. It is easy to check that the set π_D is different from π_C. However, this discrepancy is unimportant; π_D and π_C differ formally but not substantially, and neither is privileged. *It is the deep idea of π that is used in the daily reasoning of a mathematician*, who does not bother with the remote formal models π_C and π_D. What counts is the deep idea of π.

FOURTH EXAMPLE. "**Number four**" is also a deep idea. This example shows that *a deep idea may already be well formed in the mind of a child*, not necessarily gifted. We assume that this is the case *when the child can use this number* (in the context of arithmetic operations and word problems) *freely, sensibly, flexibly, understanding the meaning of what he/she is doing*. Of course, this deep idea evolves as the child gets older, but basically it remains the same idea of "four" (similarly as a growing boy remains the same person although he does not stay identical). The set $\{Ø,\{Ø\},\{Ø,\{Ø\}\},\{Ø,\{Ø\},\{Ø,\{Ø\}\}\}\}$ (or 4 in von Neumann's theory) is a formal model of "four". Another formal model of "four" is its binary representation (written, e.g., as 100_2) and so too is its Dedekind cut representation. Each of these models serves a different purpose.

FIFTH EXAMPLE. There is one single deep idea "**natural number**" and several formal models of it. The following are best known:
1) various formalizations of the Cantor-Frege approach based on the concept of one-to-one correspondence of elements of sets;

2) Peano axiom system (which may be formalized without the concept of set);
3) von Neumann's definition (mentioned before) in an axiomatic set theory.
All these three theories differ significantly. However, *a crucial criterion of acceptance of such a formal theory is its coherence with the deep idea of natural number.*

The above examples and many other examples show that *it is important not to confuse the deep ideas with their formal models*. Models depend on formalizations, which are by no means unique and may even appear artificial.

Generally one should be aware that *the deep ideas, the surface representations and the formal models can at best correspond well to each other in limited areas*. The correspondence between them is only partial. There are various exceptions to the general harmony expected between them. Some types of discrepancies will be discussed below.

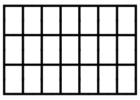
SIXTH EXAMPLE. Cauchy's definition "for every $\varepsilon>0$ there exists an N such that for all $n>N$, $| a_n - g | < \varepsilon$ " (in symbols and/or words) reduces the question of what is the **limit of a sequence** to some finite system of logical symbols and inequalities. A great achievement of 19th-century mathematics was to replace a vague notion of a limit by this clear definition and to raise considerably the standard of rigour. Put differently, *Cauchy's definition has made it possible to replace the deep idea of a limit by a surface representation*. The price paid for this is the danger that the teaching of limits may be reduced to formal transformations of inequalities and may be a cause of the regrettable fact that a deep idea of a limit may not be formed in the student's mind. Many teachers neglect the "intuition", fear that it may be misleading, and believe that intuition should play no role in the reasoning. Consequently, many students remember only those "for every $\varepsilon >0$ ...". In practice, what is assumed to be formal knowledge may degenerate into reproducing surface representations.

Similar remarks apply to the concept of the **derivative of a function**. Thurston (1994, p.163) outlined many possible ways of *thinking of* the derivative (or *conceiving of* it):
(1) *infinitesimal* (ratio of infinitesimal changes);
(2) *symbolic* (e.g., the derivative of $\sin x$ is $\cos x$, the derivative of x^n is nx^{n-1});
(3) *logical* (in terms of ε, δ);
(4) *geometric* (the slope of the tangent line, if the graph of the function has a tangent);
(5) *rate* (the instantaneous speed of $f(t)$, when t is time);
(6) *approximation* (the best linear approximation to the function near a point);
(7) *microscopic* (as if you looked under a microscope of higher and higher power).
Usually (3) is accepted as the definition; however, the deep idea includes all these features.

Some deep ideas (e.g., "smaller number", "Piaget conservation", "polygon", "fraction", "points inside a closed curve") *are formed in a person's mind* (in a long process) *before any definition is learned*. In the process of formation of other deep ideas (particularly in advanced topics) *definitions are usually learned first*. In the latter case, one may raise a pertinent question: *Is a deep idea* of a mathematical object X already *formed in the mind of a given person*? A significant criterion which may be applied to this case is: *Can this person deal with* X *freely as part of inner thinking, with understanding, correctly and without the need of referring to a definition or to a surface representation of* X? A concept, say, of $\log x$ may be acquired by somebody without becoming a deep idea; then the person has to refer to the definition (or to memorize the procedures) and to rely on transformations of surface representations. Of course, this criterion should not be interpreted mechanically; it is a clue rather than a definite requirement.

Besides the above possibilities (a deep idea prior to any definition, a deep idea after a definition), *certain well-understood concepts are used by mathematicians without any* *definition* whatsoever (possible definitions are artificial, or partially adequate, or simply superfluous). The deep idea suffices.

SEVENTH EXAMPLE. The deep idea "**two-dimensional rectangular array**" is developed in a person's mind by dealing both with real-life situations (tiles, eggs in a container) and with various mathematical schemes, visualized as in the following four examples:



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This deep idea *is not based on understanding language*. Sophisticated analysis of various aspects of children's perception of geometric and arithmetical structures of such arrays can be found in Rożek (1994). She pointed out that the concept of a rectangular 2D-array could easily be mathematized and showed some formal models of this deep idea. However, usually *no such definition is explicitly stated*. It is not needed because what is actually used is the deep idea of the array. A formal definition of a (general) double array (particularly in more complicated situations from real life) may even obscure this concept. "*When the idea is clear, the formal setup is usually unnecessary and redundant*" (Thurston, 1994, p.167).

EIGHTH EXAMPLE. The deep idea "**tetrahedron**" (tetrahedron is assumed here to be closed, that is, a solid together with its boundary) corresponds to several formal models: (a) the *set of points of a tetrahedron*; (b) the same set with *an extra structure consisting of its four faces, six edges and four vertices* ("visible attendants" in the sense of Hejný, 1993), in other words – *a geometric complex together with its combinatorial structure*; (c) *a triangular pyramid*, having a structure richer than that of a tetrahedron, with one distinguished face called "bottom"; (d) a metric space; (e) a convex set in a vector or affine space. Such different points of view were considered by Freudenthal (1991, p. 20) in his discussion of rich and poor structures in mathematics. *The deep idea of a tetrahedron implicitly contains the above aspects*; it cannot be reduced to (a) only. When we think of a tetrahedron, we automatically have in mind its basic geometric features.

NINTH EXAMPLE. There is an abundance of concepts called **angles**. They may be classified in various ways. In particular, the angles may divided into two basic types. An (N)-angle ("*number-angle*") is a *real number* (possibly a number mod2π or modπ) assigned to a geometric configuration (planar or 3D, oriented or non-oriented; this also includes angles between two curves or between a curve and a surface, or between skew half-lines) or defined in certain analytically or kinetically defined situations. An (S)-angle ("*set-angle*") is a *set* of points (i.e., a subset of the plane or 3D space) or a set of geometric figures. The measure of an (S)-angle is an (N)-angle (the converse need not be true, e.g., the (N)-angles in an n-dimensional vector space with scalar product do not correspond well to any sets of points). There is no easy, clear way of translating the deep idea of a specific kind of an (S)-angle into a rigorously formulated definition. A planar angle-region may be defined geometrically as, say, any of the two closed regions U, W determined by an unordered pair $\{H,L\}$ of half-lines (its sides) having a common end-point v. However, this approach has a weak point: a straight angle has either no vertex and no sides or infinitely many of them. Several formal models of the deep idea of such an (S)-angle are possible, e.g., (1) the set U itself, (2) the same set U with an extra structure of sides formalized, say, as $\{U, H, L\}$, (3) the set U with a distinguished vertex v, i.e., $\{U,v\}$. If an (S)-angle is defined as an ordered pair (H,L), then it determines a single angle-region only in the case of an oriented plane. In practice, *what mathematicians use in the reasoning is a* (compound) *deep idea of an angle*. It is not unlikely, though contradicting the common picture, that a university mathematician *cannot recall any definition*, though he understands very well what he is thinking about and can produce an *ad hoc* definition of an angle that fits his deep idea.

We continue showing examples of various kinds of interrelations between the three elements of the triad «deep, surface, formal», arranged so as to highlight the complexity of these interrelations. The most important assertion of this paper is that *in case of epistemological difficulties, the deep ideas prevail over the corresponding formal models*.

TENTH EXAMPLE. The standard academic presentation of the concepts: "an ordered pair" and "function", based on set theory, consists of the following steps: 1° *Kuratowski's pair* (a,b) is defined as $\{\{a\}, \{a,b\}\}$; 2° the *Cartesian product of* X and Y defined as the set $X \times Y = \{(x,y)\colon x \text{ in } X,\ y \text{ in } Y\}$; 3° a *relation* is defined as a set of pairs (i.e., any subset of the product $X \times Y$); 4° a *function* is defined as a relation satisfying the two well-known conditions; 5° a *sequence* $(a_1,..., a_n)$ is defined as a function on $\{1,...,n\}$; 6° the product $X_1 \times ... \times X_n$ is defined as the set of sequences $(x_1,...,x_n)$ such that $x_j \in X_j$ for j in $\{1,...,n\}$.

In this example a singular cognitive conflict is hidden. Namely, the above well-known presentation of six definitions is not so simple and neat as it looks. In fact, it has a serious weak point: it is easy to check that an *ordered pair* (x_1, x_2) is not the same as the sequence (x_1, x_2). Kuratowski and Mostowski (1952) commented: "*in applications usually it does not matter which of the two notions is used* ". We rephrase this statement saying: "*in the real daily work of a mathematician only the deep idea counts; formal models of a pair should exist and should have the desired properties, but they are not directly used*". We have a peculiar loop of concepts: functions are regarded as a special case of relations, relations are regarded as sets of pairs, pairs are regarded as sequences, and sequences are regarded as functions. A formal vicious circle can be avoided (see e.g. Gödel, 1940), but it nevertheless remains in various places. For example, the reader of J.L.Kelley's very popular *General Topology* (1955) may be not aware that the special case of the product $X_1 \times ... \times X_n$ for $n=2$ is not the same as the product $X_1 \times X_2$ defined earlier in the same book; in this case the discrepancy is particularly striking, because the reasoning in question is claimed by the author to be strictly rigorous (in the setting of an axiomatic system of set theory).

Although the five concepts involved in this cognitive conflict form a basis of set theory, mathematicians are not troubled by this. *What they actually use is not a formal definition but the intuitively clear deep idea of an ordered pair*. Ninety years ago one of the founders of set-theoretical mathematics wrote: "*This concept* [an ordered pair] *is fundamental to mathematics; from a psychological point of view, an ordered non-symmetric selective link of two things is primal in relation to unordered, symmetric, collective. Thinking, speaking, reading and writing are bound to temporal succession, which suggests itself before it can be passed over. A word is earlier than the set of its letters, an ordered pair* (a,b) *is earlier than an unordered pair* $\{a,b\}$" (Hausdorff, 1914, p. 32).

The four deep ideas: "ordered pair", "relation", "function", "sequence" do not form a single sequence where each next notion is defined in terms of its predecessors alone. The actual base of mathematical reasoning is the whole quartet of four deep ideas, closely tied to each other. We note that the deep idea "sequence" cannot be reduced to the correspondence $k \to a_k$. We think of $(a_1,..., a_n)$ as of terms $a_1,...,a_n$ in some *order*: a_1 first, then a_2, and so on. Yet no order is explicitly stated in the definition of a sequence as a function; it is implicit, *induced* by the natural order on $\{1,...,n\}$.

The problem of existence of two formal models: $(x_1, x_2)_{\text{pair}}$ and $(x_1, x_2)_{\text{sequence}}$ *of the deep idea* "*ordered pair*" *is markedly different from the case of two formal models of* π; the two models π_C and π_D (Example 3) are constructed in two distinct axiomatic theories, whereas *the two definitions* of $X_1 \times ... \times X_n$ *are formulated in the same theory*. We note that the case of tetrahedron (Example 8) is different from both the above case of pairs and the case of two models of π. The choice of one of possible formal models of a tetrahedron depends on *what structures of it are singled out*.

ELEVENTH EXAMPLE. **Elementary algebra** appears to be based on surface representations and formal models only. Yet, there are good reasons to believe that *algebraic deep ideas* are also formed in human minds. According to Thom (1970), "*The mathematician imparts a meaning to each* [mathematical] *sentence; this enables him to forget the place of this sentence in a any existing formalized theory. The meaning gives an ontological status to the sentence regardless of any formalization*". In particular, algebraic expressions and their transformations have various meanings, originating from their role in mathematics and elsewhere. The deep idea of an algebraic expression, e.g. $x+4$, regarded as *a distinct object*, includes its meaning, possible purposes, and its relations with other concepts (arithmetical expressions, solving equations). *A surface structure* of an algebraic expression is the arrangement of its terms and operations, e.g., $x+4$ and $4+x$ have different surface structures, though they have the same *systemic structure* (i.e., they are equal, Kieran, 1989). Both the surface structure of $x+4$ and its systemic structure are deep ideas, and so, too, are the general concept of the surface structure and the general concept of the systemic structure (they may be formed in a person's mind even though this person does not know such names and has never made such a distinction explicitly). Good command of algebraic expressions such as $a_1+...+a_n$ requires deep ideas of letter symbols, of indices, and of dots (which mean "and so on"); if a person lacks these ideas, their verbal description would be of little help. Transforming $6x+3x$ or $(-2x)\cdot 8x$ in order to "simplify the expressions" (or "performing the given operations") may be based on formal properties of the operations (e.g., distributivity). However, what is more likely is the use of informal, not explicitly specified rules, which – in the long run – may perhaps evolve into deep ideas. One may speculate that a student's procedure labelled *Automatization* in (Demby, 1997) is an indication that pertinent deep ideas are being formed in his/her mind; those students were genuinely surprised by the question "*Why do you think this is correct?*" and exclaimed, e.g.: "*It's obvious!*". A similar attitude was reported in the case of Piaget conservation (Example 2); many conserving children were surprised by questions such as "*Is the number of apples the same now?*" (they exclaimed: "*Why do you ask? Of course it's the same!*").

TWELFTH EXAMPLE. The individual deep ideas of a **straight line** in minds of Euclid, Kant and Hilbert were certainly different, but their mathematical essence (including e.g., the axiom "any two points lie on one and only one straight line") is basically the same (in spite of inevitable philosophical and cultural differences). This being so, we may speak of a single deep idea "Euclidean straight line" as an epistemological object. Metaphorically, the Nile River in the time of Euclid was different from today's Nile, but it still is the same river. Snowflakes are all different but there is a single general concept "snowflake". Similar arguments apply to many concepts, e.g., the present deep ideas "**real number**" and "**the continuity of a function of real variable**" are basically the same as those in time of Weierstrass (when they matured after very long historical process of their formation) although our present knowledge about those concepts is much richer.

Although the individual deep ideas of X in minds of specific persons need not be identical, they always have a *common core*. For example, if X denotes the expression 7(6+2), the common core includes understanding that we add 6+2=8 and then multiply 7 times 8, getting 56; if X denotes "the derivative of sine", the common core includes: the meaning of "derivative" and of "sine", the fact that the result is cosine, and the reason why that is so. Consequently, we regard the deep idea X in the mind of person A and deep idea X in the mind of B as epistemologically *the same* deep idea X (provided that both are sufficiently well formed). The experiences of thousands of people over centuries provide irrefutable *empirical* evidence of the fact that basic mathematical ideas are concordant in the above sense (in spite of numerous slips, errors and changes in the past). Hence the deep idea of X may be regarded as a single abstract epistemological object. For instance, there may be individual differences in the way people think of the number π (depending on their earlier experience and their knowledge), and yet everywhere in the world people having good command of the concept of π and its use must share some common knowledge of it (including, say, πr^2 as the area of a disc), regardless of whether the explanations are expressed in English or another language, with or without symbols.

THIRTEENTH EXAMPLE. Two separate small groups of German mathematicians working in **convexity theory** were informally asked by the author whether a vertex v of a convex polyhedron K in $\boldsymbol{R}^n$ is the same as a 0-dimensional face of K (a convex subset F of K is called a *face* of K if the conditions $x \varepsilon F$, $x = ½y+½z$, $y \varepsilon K$, $z \varepsilon K$ imply $y \varepsilon F$, $z \varepsilon F$). They answered: "Yes, of course", and appeared not much disturbed by the remark that such a face is a singleton $\{v\}$ and not just the element v. Clearly, their answers were incompatible with the set-theoretical background of convexity theory and reflected the dominance of deep ideas over formal models in their reasoning. Although the samples were not representative, this indicates a possible line of research (note that the distinction between v and $\{v\}$ was so praised by the promoters of the "new math" reforms).

FOURTEENTH EXAMPLE. The basic concepts of **category theory**, such as *category, functor, natural transformation of functors*, are deep ideas. This makes the theory robust when serious difficulties with set-theoretical foundations are dealt with (Mac Lane, 1971).

Proofs and proving. Examples 1–14 serve as illustrations of certain properties of the triad «deep, surface, formal» and of the mutual relations between its elements in various mathematical contexts. We shall now consider the concept of a proof, which is central to mathematics. Proofs play many significant roles; for a comprehensive survey, augmented with the literature of the subject, see (Hanna, 2000). The most important functions of proofs and proving are: *verification* (of the truth of a statement), *justification* and *explanation* (insight why it is true).

The general concept "proof of a theorem" is a deep idea. It is often taken for granted that the proofs found in academic books correspond well to the general description presented in books on mathematical logic. A proof of a theorem T in a given formalized axiomatic theory is (loosely speaking) a sequence of propositions T(1),...,T(n) (expressed in the language of the theory) such that T(n) is just T and each T(j) can be deduced from the axioms and the preceding propositions T(1),...,T(j – 1) by using one operation from a given list of admissible ways of inference. It is well known that (except of publications on logic) no research proofs are written in this way, which is practically unrealizable. This question is discussed in some detail in (Mac Lane, 1981) and (Davis and Hersh, 1981). It may be summarized by the following quotation from Mac Lane (in: Atiyah et al., 1994, p. 191): "*The sequence for the understanding of mathematics may be: intuition, trial, error, speculation, conjecture, proof. The mixture and the sequence of these events differ widely in different domains, but there is general agreement that the end product is rigorous proof – which we know and recognize, without the formal advice of the logicians"*. The words after the dash may be interpreted as follows: *proof* is a deep idea which is formed in a long process as a result of hard work with mathematics; the way logicians speak of proofs is valuable, but is not helpful when it is necessary to write down or verify a difficult proof (and is completely useless when a proof has not yet been conceived). Thus, formal models of proofs of typical theorems exist potentially, in highly idealized versions, but are not actually executed. In case of proofs, the distance between "deep" and "formal" seems to be greater than that in the previously considered examples. "*We should recognize that the humanly understandable and humanly checkable proofs that we actually do are what is* *most important to us, and that they are quite different from formal proofs. For the present, formal proofs are out of reach and mostly irrelevant*" (Thurston, 1994, p. 171).

FIFTEENTH EXAMPLE. In one of the school textbooks in the 1960's, the author introduced vectors and the scalar product, believing that these important concepts should offer some advantages. Indeed, they were used, in particular, to give a very short and elegant proof of the theorem of Pythagoras. Specifically, if vectors **a** and **b** forming two sides of a triangle are perpendicular, then $|\mathbf{a}–\mathbf{b}|$ is the length of the hypotenuse, which can easily be computed: $|\mathbf{a}–\mathbf{b}|^2 = (\mathbf{a}–\mathbf{b})(\mathbf{a}–\mathbf{b}) = |\mathbf{a}|^2 – 2\mathbf{ab} + |\mathbf{b}|^2$. Since the scalar product **ab** is 0, the theorem follows immediately. Nevertheless, some top students insisted later that no proof of the theorem of Pythagoras was given, although they remembered the computation. They could check the surface part, but the passage from the above equalities to the conclusion required deep ideas which were lacking in case of those students.

Actual proofs combine reasoning based on deep ideas (D) with making use of surface representations (S). Extensive use of (S) may have an adverse effect: "*the more easily recorded and communicated formalism and machinery tend to gradually take over from other modes of thinking*" (Thurston, 1994, p.167). Clearly, each proof must involve (S). This is even so when the proof is just an oral explanation that makes no use of symbols. Still, it uses words, and words are (S). When someone edits or verifies a proof and wants to *understand why* the successive steps are valid, (D) is also involved. A single step in a research proof may turn out a "big jump" if compared with single steps in a formalized theory. On the other hand, *a frequently applied and verified series of typical steps* may become a kind of "subroutine", a new "obvious step" and eventually *a new deep idea*. Only very special proofs of very special theorems are free of deep ideas, i.e., could be checked by a computer. Thus, almost all proofs involve both (D) and (S). The relations between the roles of (D) and (S) in proofs and the question which of them is dominant have been subject of many casual remarks as well as of research. A celebrated characteristic was given by Poincaré (1908, p.133): "*When a logician decomposes a proof into many elementary steps, each correct, he will not yet have the whole; this indefinable something that endows the proof with unity will escape the net*". This may be rephrased by distinguishing (a) *step by step reasoning* and (b) *comprehensive reasoning*. In a single step, (S) is indispensable and (D) is usually also needed, although such steps may often be reduced to a formal application of rules, e.g., the proof of convergence of a typical secondary school sequence can be turned into a series of routine transformations of inequalities (and then it is "rigorous"). On the other hand, with exception of trivial cases, complete understanding of the proof requires deep ideas.

SIXTEENTH EXAMPLE. In an academic textbook (Sierpiński, 1951), written by one of most famous Polish mathematicians, the proof of **the fundamental theorem of algebra** consists of two steps. First, a lemma: If f is a polynomial of degree $m\geq1$ with complex coefficients and $f(z_0)\neq0$, then there exists a complex number z_1 such that $|f(z_1)| < |f(z_0)|$. The theorem follows from the lemma by noting that the function $|f(z)|$ is continuous and must attain its minimal value at some z_0; if $f(z_0)$ were different from 0, this would contradict the lemma. I vividly remember reading this when I was a student. It was easy to verify each of some twenty steps of the proof of the lemma shown in the book, but memorizing it seemed hopeless: there were two pages of computations based on elementary properties of complex numbers. Some thirty years later I suddenly realized that the lemma was so easy that an oral proof would do. Indeed, without loss of generality we may assume that $z_0=0$ and $f(0)=1$. Suppose first that f is a polynomial of degree 1, i.e., $f(z)=1+az$. Let $z=r(\cos\varphi+i\sin\varphi)$ and let φ change from 0 to 2π. If r is fixed and small enough, $1+az$ revolves around the point 1 and for some z_1 it must be closer to 0 than the point 1. If $f(z)=1+az+a_2z^2+...$ then for sufficiently small r the higher powers a_kz^k are so small that they cannot compensate the distance resulting from $1+az$. In the case where $a_1=0$ and $f(z)=1+a_pz^p+...$ $(p>1)$ the argument is similar, $1+a_p z^p$ revolves p times. I looked again at the proof in the book: it used essentially the same argument as my "oral proof", but without reduction to easier cases and without any reference to geometry (still, it gave an explicit construction for z_1).

Sierpiński's proof – a typical manifestation of the attitude prevailing in the 1950s – is a vivid example of what Lakatos (1976, p. 142) called the *Euclidean deductivist style*. There is no hint why the argument works; *surface representations dominate* and *the role of deep ideas is minimized*. On the other hand, the above "oral version" is based on deep ideas and provides *nervus probandi* (the crucial idea of the proof which makes it valid). Sierpiński's proof is oriented towards *demonstrating the truth* while the aim of the "oral version" is twofold: *justification* as well as *better understanding why* the lemma is true.

Several authors have discussed distinctions of this kind. M. Steiner differentiates *proofs that explain* from *proofs that only demonstrate.* Wittman and Müller elaborate "content-insight proofs" which focus on the meaning. Simpson highlights "proofs through logic", which emphasize the formal, and "proofs through reasoning", which involve investigations and heuristics; for references and further details in this line, see Hanna (2000). Raman (2003) distinguishes between *private* argument (which engenders understanding) and *public* argument (with sufficient rigour for mathematical commuttnity). Raman also speaks of three types of ideas used in producing a proof. The first is called a "heuristic idea"; it is essentially private and gives a sense of understanding and a feeling that the statement ought to be true, but not conviction. The second type, called a "procedural idea", is essentially public and is based on logic and formal manipulations, which lead to a formal proof; it gives a sense of conviction, but not understanding. The third, called a "key idea", is a link between a heuristic idea and a procedural one, a "mapping" of the first to the second. If somebody has a key idea of a proof, he/she is able to see that both heuristic idea and the procedural one represent the same idea. Example 16 fits this conception perfectly; the "oral version" is a key idea and a private argument acceptable to a person with sufficiently formed relevant deep ideas that are involved in it, but for a wider audience some details must be elaborated. The contrasting pairs: "demonstrate-explain", "public-private", "procedural-heuristic" should be augmented with the pair "surface-deep".

Relations of the conception of the triad to other theories

Formation of deep ideas in the human mind. The conception of individual deep ideas is based on the assumption that they are *constructed in the minds of people*. Psychological theories like those of Piaget (Beth and Piaget, 1966; Piaget and Inhelder, 1989; Piaget and Garcia, 1989) and other authors provide some insight into the multistep process of the formation of deep ideas at various levels of the cognitive development. *An operation on deep ideas* (such as e.g. "numbers", "isometries") *may later become a deep idea at the next level*, and the process is repeated. Structures are constructed which are later structured by new structures. In particular, certain deep ideas are possible only when a suitable level of *reifications* of actions into *entities* (Kaput, 1989; Sfard, 1991) or *encapsulation* (Dubinsky, 1991) has already been attained; the deep idea of 9+24=33 cannot emerge before the person is able to grasp such equalities *proceptually* in the sense of Gray and Tall (1994); see also Gray in (Tall and Thomas, 2002, pp. 205–217).

Relations to basic philosophies of mathematics. We want the proposed theory to be philosophically as neutral as possible. *The conception of deep ideas does not require a definite philosophical commitment*. It is compatible with some forms of Platonism, with moderate constructivism ("constructivism" in the sense used in mathematics education), with moderate social constructivism, and with moderate formalism (reduced to formal models). It is our tenet that *mathematical knowledge cannot be simply transferred ready-made from the teacher to the learner and has to be actively built by the latter in his/her* *own mind*. We should draw, however, the reader's attention to the groundlessness of certain inferences. Our tenet *does not imply* that knowledge is independent of the external world. It *does not imply* that knowledge does not reflect certain timeless regularities of the world. We do not posit the primacy of the mental over the external. On the other hand, Platonism (meant as existence of mind-independent abstract objects whose properties humans attempt to discover and/or describe) *does not imply* that numbers must be identified as sets (i.e., numbers need not be identified with their formal models). It *does not imply* that discovery learning and group learning are impossible. Yet, deep ideas can be so familiar and natural to their possessor that they engender a belief of their necessity and of objective existence; this makes Platonism plausible to mathematicians. To forestall a misinterpretation, we emphasize that the conception of deep ideas is equally valid with and without Platonism, but *is hardly reconcilable with nominalism, logicism, conceptualism, intuitionism, radical apriorism, radical constructivism, radical social constructivism, and* – generally – *with those theories that are a priori dismissive and are based on denying the very foundations of rival theories*. Moreover, if Platonistic objects exist independently of human mental activity, mathematicians do not access to them by some special "intuition", but *by constructing isomorphic (or perhaps homomorphic) mental images of some of them.* We regard mathematical knowledge as the heritage of generations of creative scientists, sustained by community approval, disseminated by accepted authorities, retraced and partially reshaped by followers (Kitcher, 1983, and his "evolutionary epistemology").

Intuition. Certain features of deep ideas bring them close to other familiar conceptions. Many aspects of what we call "a deep idea" may be referred to as *intuition*. We are too ready to invoke inner intuition when no other ground of knowledge can be produced (Frege, 1884). However, *in the context of mathematics and mathematics education* this word in used in many *markedly different senses*. Apart from the interpretation of "intuition" that goes back to Descartes and Kant (non-inferential knowledge, the direct knowing without the conscious use of reasoning), several other ways of its use by mathematicians are vividly described in (Davis and Hersh, 1981). Substantially different features are attributed to intuition in (Fischbein, 1987). In (Kitcher, 1983) intuition is presented as one of most overworked terms in the philosophy of mathematics. My position is the following: Although *certain deep ideas in certain situations may be described as "intuition",* the difference between them is essential, since *deep ideas stem from conscious mathematical activities and from reasoning*. A deep idea is not "a specific mathematical intuition that is the genetic origin of concepts prior to experience". Therefore it is best *to separate the clear conception of a deep idea from the many confusing usages of the word "intuition"*. Moreover, the popular stereotypical image "*formal mathematics versus intuition*" is a false dichotomy, a dichotomy that results from distorted perspective.

Meaning. Following the arguments of Thom (1970, 1972) and Skemp (1982) we may try to describe the essence of the deep idea of an object X as *the meaning* of X. Then, however, the question arises of how to describe "meaning". Mathematicians either regard the word "meaning" as a non-technical informal word of ordinary language or reduce it to some *definiens*, e.g., by saying that the meaning of "square" is "rectangle with equal sides" (this property is, of course, part of the deep idea "square", which embraces much more). It is well known that no satisfactory theory of "meaning" or "sense" can be found in texts on philosophy, logic, semiotics or linguistics (Quine, 1980, Essays II, III, VII; Sierpińska, 1994). If the meaning of an expression is understood as "the idea expressed by it" (Quine, 1980, Essay III), then explaining deep ideas by meaning is getting us nowhere. According to various theories, a mathematical object X may have a well-determined pre-existing meaning that we study and describe, or the meaning of X is something constructed by us in our minds, or the meaning of X is a certain way of understanding X. Let us also note that according to Cobuild English Dictionary, "the meaning of a word, expression or gesture is the thing or idea that it refers to or represents and which can be explained using other words". While accepting the first part of this text, we must stress that in many cases the meaning of a mathematical object (as we interpret it) cannot be fully explained by words.

Concept. One may argue that the deep idea of an object X (such as, say, √2) is the same as the concept of X, and hence the term "deep idea" is superfluous. Yet, there are essential differences between "the concept of √2" and "the deep idea of √2". Another reason why we need a separate term "deep idea" is pragmatic: in the literature, the term "concept" is assigned various incompatible senses. Samples: in logic, concept may be "*the meaning of a name*" (Ajdukiewicz, 1974; Sierpińska, 1994, p.42); in philosophy, concept may be "*a word which has a general meaning; knowledge of a concept is what enables to define a word*" (Vesey and Foulkes, 1990). According to Freudenthal (1991, p.18) "*Concept of X seems to mean how one conceives of an object X in a certain perspective, say, by inspection, reflection, analysis, scrutiny, or whichever you wish*", "*What is the difference between number and number concept* (...), *between X (an object) and the concept of X?* (...) *There is at any rate a difference between both of them.*". His statements would please a Platonist, since they distinguish between an *object* (e.g., number) and the *concept of it* in human minds, which makes this object external to human thought. To complete the picture, we note that the word "concept" is never used in (Beth and Piaget, 1966) except for quotations from other authors.

Mental images. Individual deep ideas have certain features of *mental images* (in the sense of Tall and Vinner, 1981), of *mental objects* (Freudenthal, 1991, p. 18), and of *Vorstellungen* (internal representations, Meissner, 2002; Goldin, 2002). Although generally the individual deep idea of X includes the mental image of X, *a person may have a transient private mental image* of, say, log*x*, and yet the understanding of log*x* need not be adequate and robust, and hence *this concept may not be yet a deep idea*.

Instrumental and relational understanding. Skemp distinguished between *instrumental understanding* (choosing and applying rules without knowing why) and *relational understanding* (knowing both: what to do and why); see (Tall and Thomas, 2002). The former means restricting the task to surface representations while the latter either involves
5 relevant deep ideas or paves the way for the emergence of deep ideas in the future.

Conclusion

• The central tenet of the proposed theory (its hard core in the sense of Lakatos) is that the triad «**deep**, **surface**, **formal**» provides an adequate framework to work on *the nature of mathematics as a body of knowledge*. Formal logic alone is not sufficient to explain some very basic facts concerning the reasoning of mathematicians (even if the analysis is confined to the final product of it, available in a published form), because it requires a prior setting of a precise admissible language. Therefore *formalized reasoning is restricted to surface representations only* and cannot fully explain some fundamental features of mathematical objects. **Formal models** are important as tools of global justification and may help us to avoid being misled by intuition. They are indispensable in the case of more advanced deep ideas, ideas which cannot be simply abstracted from activities involving real life objects.

• **Surface representations** are not only means to communicate mathematics, but also invaluable tools for mathematical reasoning and computations, and are instrumental in forming and developing concepts.

• The features of **deep ideas** are described in the present paper in the context of pertinent examples. *Most of mathematical reasoning is controlled by the deep ideas*, which prevail over the corresponding formal models in case of a cognitive conflict. *Deep ideas originate from conscious mathematical activities and reasoning in various situations from real life, science, and mathematics itself*. They form a complex web of concepts linked by a whole host of types of meaning-based relationships (which depend on a wide variety of types of activities of their origins), described only partially in the literature (for a study of relationships such as "*the same*" and "*can be identified*" see Semadeni, 2002b). In the process of historical development, *after having reached a certain level of maturity, deep ideas keep their identity*. In spite of (a) differences between individual deep ideas and (b) the changes due to the evolution of mathematics, there is a common core and no essential ambiguity concerning the most basic properties of the notions involved in any deep idea once it has sufficiently matured. Languages are different but the individual deep ideas are concordant. This is why mathematics is universal.

• Hopefully, *the conception of deep ideas may act as a bridge between the Platonist attitude of mathematicians and the constructivist trends among researchers in mathematics education*, and hence it may help to reconcile these divergent positions, easing the problems mentioned in the Introduction.

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